4 Balance Laws in Continuum Mechanics

4.1 Balance of Mass

Let $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$ be a motion and $\rho_0(\mathbf{X})$ be the density in the initial configuration Ω (at time t=0). Let $\rho(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega_t$ be the density in the configuration at time t.

Now let $B_t = \boldsymbol{\phi}(B, t)$ be a sub-body of Ω_t. Then, by the change of variables formula, we have:

$$\text{mass in } B_t \;= \int_{B_t} \rho(\mathbf{x}, t)\, dV_{\mathbf{x}} = \int_B \rho(\boldsymbol{\phi}(\mathbf{X}, t), t) \det D\boldsymbol{\phi}(\mathbf{X}, t)\, dV_{\mathbf{X}} \;.$$

By conservation of mass, this should equal $\int_B \rho_0(\mathbf{X})\, dV_{\mathbf{X}}$. Since $B \subset \Omega$ is arbitrary, it follows from Theorem 3.1 that

$$\rho_0(\mathbf{X}) = \rho(\boldsymbol{\phi}(\mathbf{X}, t), t) \det D\boldsymbol{\phi}(\mathbf{X}, t), \quad \forall \mathbf{x} \in \Omega \iff \rho(\mathbf{x}, t)|_{\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t)} = \frac{\rho_0(\mathbf{X})}{\det D\boldsymbol{\phi}(\mathbf{X}, t)}$$

$$\iff \rho(\mathbf{x}, t) = \left.\frac{\rho_0(\mathbf{X})}{\det D\boldsymbol{\phi}(\mathbf{X}, t)}\right|_{\mathbf{X} = \boldsymbol{\psi}(\mathbf{x}, t)}, \; \forall \mathbf{x} \in \Omega_t. \tag{4.1}$$

where $\boldsymbol{\psi}(\mathbf{x}, t)$ is the inverse of the map $\boldsymbol{\phi}(\mathbf{X}, t)$. The above is the balance of mass condition expressed in *material coordinates.*

Remark 4.1 (Balance of mass condition in spatial coordinates). *As noted in Remark 3.3, we have balance (conservation) of mass if the density* $\rho(\mathbf{x}, t)$ *satisfies*

$$\frac{d}{dt} \int_{B_t} \rho(\mathbf{x}, t)\, dv = 0$$

for all sub-bodies $B_t \subset \Omega_t$*. By the transport theorem and the localisation principle, it follows that*

$$\frac{D}{Dt}\rho + \rho(\nabla \cdot \mathbf{v}) = 0 \iff \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot (\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) = 0, \tag{4.2}$$

which is the continuity equation.

The conservation of mass (4.1) can be used to give an alternative proof of version 2 of the transport theorem (Theorem 3.4).

Example 4.2. *[Alternative proof of version 2 of Transport Theorem]*

Let $\boldsymbol{\phi}(\mathbf{x}, t)$ *be a motion,* $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$*, and let* $\rho_0(\mathbf{X})$ *be the density in the initial configuration and* $\rho(\mathbf{x}, t)$ *be the density for* $\mathbf{x} \in \Omega_t$ *(configuration at time* t*). If the conservation of mass holds, then given any* $f(\mathbf{x}, t)$ *defined for* $\mathbf{x} \in \Omega_t$*, we have:*

$$\frac{d}{dt}\left(\int_{B_t} f(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV_{\mathbf{x}}\right) = \int_{B_t} \left(\frac{D}{Dt} f(\mathbf{x}, t)\right) \rho(\mathbf{x}, t)\, dV_{\mathbf{x}}$$

where $B \subset \Omega$ *and* $B_t = \boldsymbol{\phi}(B, t)$*.*

Proof. Use (4.1) and the change of variables formula for multiple integrals. □

4.2 Forces Acting on and within a Continuum

Internal Forces / Stresses

Let $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$ be a motion, then the stress principle / hypothesis of Cauchy / Euler states that for any fixed t, there exists a vector field $\mathbf{t} : \Omega_t \times S^2 \to \mathbb{R}^3$ where $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ such that for any $\mathbf{y} \in \Omega_t$ and for any smooth surface $\tilde{S}$ in Ω_t passing through $\mathbf{y}$ with unit normal $\mathbf{n}$ at $\mathbf{y}$, the (internal) force per unit area at $\mathbf{y}$ exerted by the material on one side of $\tilde{S}$ (the side into which $\mathbf{n}$ is pointing) on the material on the other side is $\mathbf{t}(\mathbf{y}, \mathbf{n})$. The vector field $\mathbf{t}$ is called the *Cauchy stress vector.*

Remark 4.3. *We will prove in section 4.3 that it is a consequence of the stress principle of Cauchy/Euler (i.e. that the Cauchy stress vector is given by* $\mathbf{t} = (t_i(\mathbf{x}, \mathbf{n}))$ *on any surface through* $\mathbf{x}$ *with unit normal* $\mathbf{n}$*) that there exists a cartesian tensor with components* $T_{ij}(\mathbf{x})$ *called the* Cauchy stress tensor $\underline{\underline{\mathbf{T}}}(\mathbf{x})$ *such that the stress vector* $\mathbf{t}$ *can be expressed as*

$$t_i(\mathbf{x}, \mathbf{n}) = T_{ij}(\mathbf{x})n_j, \tag{4.3}$$

which we write as

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = T(\mathbf{x})\mathbf{n}, \quad \forall \mathbf{x} \in \Omega \tag{4.4}$$

for all unit vectors $\mathbf{n}$*.*

External Forces

External forces acting on a continuum body are of two kinds.

1. Body forces (e.g. gravity) provide a force per unit mass acting on the body. Other examples include forces induced by electric/magnetic fields. The body force is described by a vector field $\mathbf{b}(\mathbf{y},t),\ \mathbf{y}\in\Omega_t,\ \mathbf{b}(\cdot,t):\Omega_t\to\mathbb{R}^3$ which gives the force per unit mass at the point $\mathbf{y}\in\Omega_t$ at the time t. The total body force at time t is:

$$\begin{aligned}\int_{\Omega_t}\mathbf{b}(\mathbf{y},t)\rho(\mathbf{y},t)\,dV_{\mathbf{y}} &= \int_{\Omega}\mathbf{b}(\boldsymbol{\phi}(\mathbf{X},t),t)\frac{\rho_0(\mathbf{X})}{\det(D\boldsymbol{\phi}(\mathbf{X},t))}\det(D\boldsymbol{\phi}(\mathbf{X},t))\,dV_{\mathbf{X}}\\ &= \int_{\Omega}\mathbf{b}(\boldsymbol{\phi}(\mathbf{X},t),t)\rho_0(\mathbf{X})\,dV_{\mathbf{X}}\end{aligned}$$

   Gravity corresponds to $\mathbf{b}=-g\mathbf{e}_3$ where g is the acceleration due to gravity.

2. External tractions are a force per unit area acting on the boundary $\partial\Omega_t$ of the body given by $\mathbf{g}(\mathbf{y},t)$ for $\mathbf{y}\in\partial\Omega_t$. This gives rise to a total boundary force $\int_{\partial\Omega_t}\mathbf{g}(\mathbf{y},t)dA_{\mathbf{y}}$.

Example

$\mathbf{g}(\mathbf{y})=-p\mathbf{n}(\mathbf{y})$ (push inwards on boundary), $\mathbf{y}\in\partial\Omega_t$ where $\mathbf{n}(\mathbf{y})$ is the outward unit normal to $\partial\Omega_t$ at $\mathbf{y}$, corresponds to an externally imposed pressure of magnitude $p>0$.

Balance Laws

The continuum within a volume $\tilde{B}$ is in equilibrium when the total force acting on it is zero and the total moment of the forces is also zero. When it is not in equilibrium, the rate of change of (linear) momentum equals the total applied force and the rate of change of angular momentum equals the total moment of the applied forces.

- (Linear) Momentum:
$$\int_{\tilde{B}}\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\,dV_{\mathbf{x}}$$
- Angular Momentum:
$$\int_{\tilde{B}}\mathbf{x}\times\mathbf{v}(\mathbf{x},t)\rho(\mathbf{x},t)\,dV_{\mathbf{x}}$$

Let $\mathbf{b}$ be the body force per unit mass acting on $\tilde{B}$ and $\mathbf{t}$ be the boundary traction per unit area.

- Total Force:
$$\int_{\tilde{B}}\mathbf{b}(\mathbf{x},t)\rho(\mathbf{x},t)\,dV_{\mathbf{x}}+\int_{\partial\tilde{B}}\mathbf{t}(\mathbf{x},t)\,dA_{\mathbf{x}}$$
- Total Moment:
$$\int_{\tilde{B}}\mathbf{x}\times\mathbf{b}(\mathbf{x},t)\rho(\mathbf{x},t)\,dV_{\mathbf{x}}+\int_{\partial\tilde{B}}\mathbf{x}\times\mathbf{t}(\mathbf{x},t)\,dA_{\mathbf{x}}$$

Balance of Linear Momentum

For any $\tilde{B}_t \subset \Omega_t$:

$$\frac{d}{dt}\left(\int_{\tilde{B}_t} \mathbf{v}(\mathbf{x},t)\rho(\mathbf{x},t)\, dV_{\mathbf{x}}\right) = \int_{\tilde{B}_t} \mathbf{b}(\mathbf{x},t)\rho(\mathbf{x},t)\, dV_{\mathbf{x}} + \int_{\partial\tilde{B}_t} \mathbf{t}(\mathbf{x},t) dA_{\mathbf{x}}$$

and using Theorem 3.4 we obtain

$$\begin{aligned} &\int_{\tilde{B}_t} \left(\frac{D}{Dt}\mathbf{v}(\mathbf{x},t)\right)\rho(\mathbf{x},t)\, dV_{\mathbf{x}} = \int_{\tilde{B}_t} \mathbf{b}\rho\, dV_{\mathbf{x}} + \int_{\partial\tilde{B}_t} \mathbf{t}\, dA_{\mathbf{x}} \\ \iff &\int_{\tilde{B}_t} \left[\left(\frac{D}{Dt}\mathbf{v}\right) - \mathbf{b}\right]\rho\, dV_{\mathbf{x}} = \int_{\partial\tilde{B}_t} \mathbf{t}\, dA_{\mathbf{x}} \qquad (4.5) \end{aligned}$$

Balance of Angular Momentum

For any $\tilde{B}_t \subset \Omega_t$:

$$\frac{d}{dt}\left(\int_{\tilde{B}_t} \mathbf{x}\times\mathbf{v}(\mathbf{x},t)\rho(\mathbf{x},t)\, dV_{\mathbf{x}}\right) = \int_{\tilde{B}_t} \mathbf{x}\times\mathbf{b}(\mathbf{x},t)\rho(\mathbf{x},t)\, dV_{\mathbf{x}} + \int_{\partial\tilde{B}_t} \mathbf{x}\times\mathbf{t}(\mathbf{x},t)\, dA_{\mathbf{x}}$$

and using Theorem 3.4 we obtain

$$\int_{\tilde{B}_t} \left[\left(\frac{D}{Dt}\mathbf{x}\times\mathbf{v}(\mathbf{x},t)\right) + \mathbf{x}\times\left(\frac{D}{Dt}\mathbf{v}(\mathbf{x},t)\right)\right]\rho(\mathbf{x},t)\, dV_{\mathbf{x}} = \int_{\tilde{B}_t} \mathbf{x}\times\mathbf{b}\rho\, dV_{\mathbf{x}} + \int_{\partial\tilde{B}_t} \mathbf{x}\times\mathbf{t}\, dA_{\mathbf{x}} \qquad (4.6)$$

$$\iff \int_{\tilde{B}_t} \mathbf{x}\times\left[\left(\frac{D}{Dt}\mathbf{v}\right) - \mathbf{b}\right]\rho\, dV_{\mathbf{x}} = \int_{\partial\tilde{B}_t} \mathbf{x}\times\mathbf{t}\, dA_{\mathbf{x}}. \qquad (4.7)$$

4.3 The Cauchy Stress Tensor

In this section we work in spatial coordinates and for convenience/clarity we suppress the time dependence. We will prove that it is a consequence of the stress principle of Cauchy/Euler (i.e. that the Cauchy stress vector is given by $\mathbf{t} = (t_i(\mathbf{x}, \mathbf{n}))$ on any surface through $\mathbf{x}$ with unit normal $\mathbf{n}$) that there exists a cartesian tensor with components $T_{ij}(\mathbf{x})$ called the *Cauchy stress tensor* $\underline{\underline{\mathbf{T}}}(\mathbf{x})$ such that the stress vector $\mathbf{t}$ can be expressed as

$$t_i(\mathbf{x}, \mathbf{n}) = T_{ij}(\mathbf{x})n_j, \tag{4.8}$$

which we write as

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = T(\mathbf{x})\mathbf{n}, \quad \forall \mathbf{x} \in \Omega \tag{4.9}$$

for all unit vectors $\mathbf{n}$.

Let $\mathbf{x}_0 \in \Omega$ and $\tilde{\mathbf{n}} = (\tilde{n}_i)$ be a unit vector with $\tilde{n}_i > 0$. Define $\tilde{B}$ to be the right sided tetrahedron with three sides S_1, S_2, S_3 (each with outward unit normal $-\mathbf{e}_i$ and area Δ_i) parallel to the coordinate hyperplanes $x_i = 0$, with its right angled vertex at $\mathbf{x}_0$, and let $\tilde{\mathbf{n}}$ be the outward unit normal to the fourth side of the tetrahedron, labelled S (of area Δ). We next consider linearly scaling the tetrahedron $\tilde{B}$ about $\mathbf{x}_0$ by a factor $\epsilon > 0$. Then we have:

$$\begin{aligned}\int_{\partial \tilde{B}} \mathbf{t}(\mathbf{x}, \mathbf{n}(\mathbf{x}))dA_{\mathbf{x}} &= \int_S \mathbf{t}(\mathbf{x}, \tilde{\mathbf{n}})dA_{\mathbf{x}} + \sum_{i=1}^{3} \int_{S_i} \mathbf{t}(\mathbf{x}, -\mathbf{e}_i)dA_{\mathbf{x}} \\ &\approx \mathbf{t}(\mathbf{x}_0, \tilde{\mathbf{n}})\Delta + \sum_{i=1}^{3} \mathbf{t}(\mathbf{x}_0, -\mathbf{e}_i)\Delta_i \quad \text{for small } \epsilon > 0\end{aligned} \tag{4.10}$$

Lemma

By the divergence theorem

$$\mathbf{0} = \int_{\partial \tilde{B}} \mathbf{n}(\mathbf{x})dA_{\mathbf{x}} = \tilde{\mathbf{n}}\Delta - \mathbf{e}_1\Delta_1 - \mathbf{e}_2\Delta_2 - \mathbf{e}_3\Delta_3$$

Hence, dotting with $\mathbf{e}_i$ yields

$$\Delta_i = (\tilde{\mathbf{n}} \cdot \mathbf{e}_i)\Delta = \tilde{n}_i\Delta.$$

Applying the lemma to (4.10) and using (4.5) we obtain:

$$\Delta\left[\mathbf{t}(\mathbf{x}_0,\tilde{\mathbf{n}})+\sum_{i=1}^{3}\mathbf{t}(\mathbf{x}_0,-\mathbf{e}_i)\tilde{n}_i)\right]\approx\int_{\partial\tilde{B}}\mathbf{t}\,dA_{\mathbf{x}}=\int_{\tilde{B}}\rho\left[\frac{D\mathbf{v}}{Dt}-\mathbf{b}\right]dV_{\mathbf{x}}$$
$$\approx\mathbf{f}(\mathbf{x}_0)\text{ vol. }\tilde{B}_t,\tag{4.11}$$

where $\mathbf{f}(\mathbf{x})=\rho\left[\frac{D\mathbf{v}}{Dt}-\mathbf{b}\right]$. Next note that $\frac{\text{vol }\tilde{B}}{\Delta}\to 0$ as $\epsilon\to 0$ (note Δ is proportional to area $\partial\tilde{B}$). Hence dividing (4.11) by Δ and letting $\epsilon\to 0$ yields:

$$\mathbf{0}=\mathbf{t}(\mathbf{x}_0,\tilde{\mathbf{n}})+\sum_{i=1}^{3}\mathbf{t}(\mathbf{x}_0,-\mathbf{e}_i)\tilde{n}_i\iff\mathbf{t}(\mathbf{x}_0,\tilde{\mathbf{n}})=-\sum_{i=1}^{3}\mathbf{t}(\mathbf{x}_0,-\mathbf{e}_i)\tilde{n}_i\qquad(**)$$

Next note, by continuity, allowing $\tilde{\mathbf{n}}\to\mathbf{e}_i$ in (**) yields $\mathbf{t}(\mathbf{x}_0,-\mathbf{e}_i)=-\mathbf{t}(\mathbf{x}_0,\mathbf{e}_i)$. Thus

$$\mathbf{t}(\mathbf{x}_0,\tilde{\mathbf{n}})=\sum_{i=1}^{3}\mathbf{t}(\mathbf{x}_0,\mathbf{e}_i)\tilde{n}_i.$$

Define $T_{ij}(\mathbf{x}_0)=t_i(\mathbf{x}_0,\mathbf{e}_j)$, then

$$t_i(\mathbf{x}_0,\tilde{\mathbf{n}})=T_{ij}(\mathbf{x}_0)\tilde{n}_j\iff\mathbf{t}(\mathbf{x}_0,\tilde{\mathbf{n}})=T(\mathbf{x}_0)\tilde{\mathbf{n}}.\tag{4.12}$$

Note: Different choices of the signs of $\tilde{n}_i$ are dealt with by minor modifications of the above argument.

Example

Consider a planar surface with normal $\mathbf{e}_i$. Then $t_j=T_{ji}$ on the surface.

In this case, T_{ii} (no sum) is called a *normal stress* and $T_{ij}, i\neq j$ are called *shear stresses* on the surface.

4.4 Equations of Motion

It follows from the balance of linear momentum and the properties of the Cauchy stress tensor (4.9) that for any $\tilde{B}_t\subset\Omega_t$, (4.5)

$$\int_{\tilde{B}_t}\rho\left[\frac{Dv_i}{Dt}-b\right]dV_{\mathbf{x}}=\int_{\partial\tilde{B}_t}T_{ij}n_j\,dA_{\mathbf{x}},\quad\forall\tilde{B}_t\subset\Omega_t$$

By the divergence theorem it now follows that:

$$\int_{\tilde{B}_t}\rho\left[\frac{Dv_i}{Dt}-b_i\right]-\frac{\partial}{\partial x_j}T_{ij}\,dV_{\mathbf{x}}=0,\quad\forall\tilde{B}_t\subset\Omega_t$$

By the localisation Theorem (3.1) it now follows that the integrand is zero and hence

$$\rho(\mathbf{x},t)\frac{Dv_i}{Dt}(\mathbf{x},t)=\rho(\mathbf{x},t)b_i(\mathbf{x},t)+\frac{\partial}{\partial x_j}T_{ij}(\mathbf{x},t),\quad\forall\mathbf{x}\in\Omega_t$$

or, equivalently,

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial}{\partial x_k}v_i\right)=\rho b_i+\frac{\partial}{\partial x_j}T_{ij}. \tag{4.13}$$

From balance of Angular momentum (4.16) and (4.9) it follows that

$$\int_{\tilde{B}_t}\mathbf{x}\times\left[\frac{D\mathbf{v}}{Dt}-\mathbf{b}\right]\rho dv_{\mathbf{x}}=\int_{\partial\tilde{B}_t}\overbrace{\mathbf{x}\times(T\mathbf{n})}^{\epsilon_{ijk}x_j(T_{kl}n_l)}\,dA_{\mathbf{x}}.$$

Hence, by the divergence theorem,

$$\int_{\partial\tilde{B}_t}\frac{\partial}{\partial x_l}(\epsilon_{ijk}x_jT_{kl})-\epsilon_{ijk}x_j\left[\frac{Dv_k}{Dt}-b_k\right]\rho\,dV_{\mathbf{x}}=0$$

$$\iff\int_{\partial\tilde{B}_t}\epsilon_{ijk}x_j\quad\underbrace{\left[-\rho\frac{Dv_k}{Dt}+\rho b_k+\frac{\partial}{\partial x_l}T_{kl}\right]}_{=0\text{ by linear momentum equations (4.13)}}\quad+\epsilon_{ilk}T_{kl}=0$$

Hence,

$$\int_{\partial\tilde{B}_t}\epsilon_{ilk}T_{kl}\,dV_{\mathbf{x}}=0,\quad\forall\ \tilde{B}_t\subset\Omega_t$$

By the localisation theorem 3.1, $\epsilon_{ilk}T_{kl}=0$ in Ω_t. Hence

$$\begin{aligned}0&=\epsilon_{imn}\epsilon_{ilk}T_{kl}=[\delta_{ml}\delta_{nk}-\delta_{mk}\delta_{ln}]T_{kl}\\&=T_{nm}-T_{mn}\Rightarrow T_{nm}=T_{mn}.\end{aligned} \tag{4.14}$$

and the Cauchy stress tensor is symmetric.

In summary we have, by (4.13), (4.14), (4.2), the following equations of motion:

$$\boxed{\text{Linear Momentum Balance: }\rho\frac{Dv_i}{Dt}=\rho b_i+\frac{\partial T_{ij}}{\partial x_j}} \tag{4.15}$$

$$\boxed{\text{Angular Momentum Balance: }T_{ij}=T_{ji}} \tag{4.16}$$

$$\boxed{\text{Mass Balance: }\frac{D\rho}{Dt}+\rho(\nabla\cdot\mathbf{v})=0\iff\frac{\partial\rho}{\partial t}+\nabla\cdot(\mathbf{v}\rho)=0} \tag{4.17}$$

Solving the equations

The derived equations are valid for any continuum. In order to solve for a particular continuum, we require constitutive relations relating T_{ij} to other variables e.g. $T_{ij}=T_{ij}$ $(\mathbf{x},\rho,\mathbf{v},\frac{\partial\phi_i}{\partial X_\alpha},...)$. The particular form will depend on the material (however symmetric of T_{ij} is necessary). Any particular problem also requires the specification of boundary conditions, typically we specify the velocity or displacement or traction (or a combination).

Examples

1. For an inviscid (ideal) fluid $T_{ij} = -P\delta_{ij}$ where $P = P(\mathbf{x}, t)$ is the pressure, so $\frac{\partial}{\partial x_j} T_{ij} = -\frac{\partial P}{\partial x_j}$ yielding the Euler equations

$$\rho \frac{Dv_i}{Dt} = \rho b_i - \frac{\partial P}{\partial x_i} \tag{4.18}$$

The incompressibility condition becomes

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = 0. \tag{4.19}$$

A typical boundary condition in solving the Euler equations is the "no normal flow" condition that $\mathbf{v}.\mathbf{n} = 0$ at a rigid boundary.

2. For an compressible fluid $T_{ij} = -P\delta_{ij}$ where $P = \Pi(\rho)$ is the pressure, where Π is the pressure as a function of density, typically with $\Pi' > 0$ (pressure increases with density)

3. Equations of motion for an incompressible Newtonian (viscous) fluid: here we assume that

$$T_{ij} = -P\delta_{ij} + 2\mu S_{ij}, \tag{4.20}$$

where $S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ are the components of the rate of stretch tensor, and μ is called the viscosity[5] (the kinematic viscosity is $\nu = \frac{\mu}{\rho}$). Hence,

$$\begin{aligned}\frac{\partial}{\partial x_j} T_{ij} &= -\frac{\partial P}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ &= -\frac{\partial P}{\partial x_i} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)\right) \\ &= -\frac{\partial P}{\partial x_i} + \mu \Delta v_i\end{aligned}$$

With this form for the stress tensor, the equations of motion are

$$\rho \frac{Dv_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \Delta v_i + \rho b_i$$

together with the incompressibility condition

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = 0$$

which is the mass balance condition. This system is known as the Navier-Stokes Equations.

A typical boundary condition imposed when solving this system is the "no slip" boundary condition that $\mathbf{v} = \mathbf{0}$ at a rigid boundary.

Remark 4.4. *Recall that for an incompressible fluid* $\nabla \cdot \mathbf{v} = 0$ *since all admissible motions satisfy* $\det(D\boldsymbol{\phi}(\mathbf{x}, t)) = 1$. *In this case* $\rho(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{X},t)} = \frac{\rho_0(X)}{1}$ *and so the continuity/mass balance equation* (4.17) *is automatically satisfied. In this course we work mainly with incompressible fluids so motions satisfy* $\det(D\boldsymbol{\phi}(\mathbf{X}, t)) \equiv 1$.

For all admissible motions $\frac{\partial v_i}{\partial x_i}(\mathbf{x}, t) = 0 \iff \rho(\boldsymbol{\phi}(\mathbf{X}, t), t) = \rho_0(\mathbf{X})$ *and we usually take* $\rho_0(\mathbf{X}) =$ *constant* $= \rho_0$ *(uniform density). (See sheet 3 Q3.)*

[5] The coefficient μ is the viscosity which dimensionally has the units of stress $\times$ time (e.g. Pascals $\times$ Seconds, 1Pascal $= \frac{1N}{m^2}$).

Note: for conservative body forces (e.g., gravity) $\rho b_i = \frac{\partial \psi}{\partial x_i}$ and the force term can be absorbed into the pressure term by modifying it to $\tilde{P} = P - \psi$.

Definition 4.5. *For fluids, a flow is* ***steady*** *if* $\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x}, t) = \mathbf{0}$. *In this case, the equations of linear momentum balance* (4.15) *are*

$$\rho \left(v_k \frac{\partial}{\partial x_k} \right) v_i = -\rho b_i + \frac{\partial T_{ij}}{\partial x_j}.$$

Definition 4.6. *A continuum is in* ***equilibrium*** *if* $\mathbf{v} = 0$.

In this case, the equations of linear momentum balance (4.15) *yield*

$$\rho b_i + \frac{\partial}{\partial x_j} T_{ij} = 0.$$

Remark 4.7. *The force exerted by a continuum on a body* B *contained within it (e.g, a body immersed in a fluid) can be determined by integrating the Cauchy stress vector over the boundary of the body. Hence, this force is given by*

$$\int_{\partial B} \mathbf{t}\, dA = \int_{\partial B} \mathbf{T}\mathbf{n}\, dA.$$

5 Properties of solutions of the Euler Equations

The main example of a continuum theory which we study in this course will be the Euler equations for flow of an incompressible, ideal fluid:

$$\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla P + \rho_0 \mathbf{F} \quad . \tag{5.1}$$

We assume that the body force is conservative so that $\mathbf{F} = \nabla \psi$. The conservation of mass equation (3.4) then reduces to the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega. \tag{5.2}$$

Recall the important concept in the study of fluid flows of the *vorticity* $\boldsymbol{\omega}(\mathbf{x}, t) = (\omega_i(\mathbf{x}, t))$, defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{5.3}$$

This is a measure of the rotation inherent in the flow. We say that the flow is *irrotational* if the vorticity (5.3) is identically zero.

5.1 Bernoulli's Theorem

Theorem 5.1. *Let* $\mathbf{v} = \mathbf{v}(\mathbf{x})$ *be a steady solution of the incompressible Euler equations and let the body force satisfy* $\boldsymbol{b} = \nabla \chi$. *Then*

$$H = \left(\frac{P}{\rho_0} + \chi + \frac{1}{2}|\mathbf{v}|^2 \right)$$

is constant along streamlines of the flow.